Emittance oscillation in the drift space of split photoinjectors

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Emittance oscillation in the drift of split photoinjectors has a "double-minimum" feature, which has been used for optimizing the working point of split photoinjectors. To better understand this feature, approximate emittance expression is derived based on beam spreading and compared with simulations. A general formula for the perturbative rms emittance calculation was derived.

I. INTRODUCTION

In split photoinjectors there is a long drift between a short gun and a booster. The beam out of the gun is focused. It reaches a waist in the drift and then diverges due to space-charge defocusing. During the optimization of the Linac Coherent Light Source (LCLS) photoinjector, it was found that the emittance has an interesting "double minimum" feature in the drift, while the local emittance maximum more or less coincides with the beam waist [1]. This special location has been chosen as the new working point for the LCLS photoinjector, where the beam is matched into a booster, shifting the second emittance minimum to a sufficiently high energy.

There were some efforts to explain such an emittance oscillation in drift space based on the familiar emittance oscillation around the invariant envelope [2, 3]. However, the existence of invariant envelopes requires external focusing to balance the space-charge defocusing. Thus, in free-drift regions, there is no invariant envelope and an alternative treatment is needed. In this note, I will show that the main feature of emittance oscillation in the drift space can be explained by beam spreading of individual longitudinal slices in free space.

In order to compute the emittance analytically, a general perturbative emittance formula is derived, which should be useful for other applications involving the emittance calculation.

II. REVIEW OF BEAM SPREADING DUE TO SPACE CHARGE

In drift regions, the transverse envelope of a cylindrical beam is governed by the simple envelope equation [4, 5]

$$\hat{\sigma}_r'' - \frac{\kappa_s}{\beta_r^2\gamma_r^2}\,\frac{1}{\hat{\sigma}_r} = 0, \tag{1}$$

where $\hat{\sigma}_r = \sqrt{\beta_r\gamma_r}\sigma_r$ is the reduced beam envelope, κ_s is the beam perveance, and $\beta_r\gamma_r$ is the relativistic factor of the reference particle. A prime symbol means differentiation with respect to the longitudinal coordinate s.

Assuming negligible perveance variations, the envelope equation can be normalized into a universal form

$$\tau'' - \frac{1}{\tau} = 0, \quad \text{with} \quad \tau = \frac{\hat{\sigma}_r}{\sqrt{\kappa_s}/\beta_r\gamma_r}. \tag{2}$$

This equation can be obtained from the Hamiltonian $H = p_\tau^2/2 - \ln\tau$, which obviously is a constant of motion and thus yields

$$\tau'^2 = \tau_0'^2 + 2\ln\frac{\tau}{\tau_0}, \tag{3}$$

where τ_0 and τ_0' are the initial values. The envelope can thus be written as

$$\tau = \tau_0\,\exp\!\left(\frac{\tau'^2 - \tau_0'^2}{2}\right) = \tau_w\,e^{\tau'^2/2}, \tag{4}$$

which grows symmetrically away from a waist $\tau_w = \tau_0 e^{-\tau_0'^2/2}$ located at $\tau' = 0$. Equation (3) can be further integrated, using the waist as the origin for simplicity, as

$$\frac{\Delta s}{\sqrt{2}\,\tau_w} = \pm\int_1^{\frac{\tau}{\tau_w}} \frac{dx}{2\sqrt{\ln x}}, \tag{5}$$

which can not be expressed with elementary functions. Approximate expressions can be obtained by expanding $1/\sqrt{\ln x}$ around $x = 1$ and by term-by-term integration, which gives $\sqrt{\tau/\tau_w - 1} + (1/12)\,(\tau/\tau_w - 1)^{3/2} + \cdots$. The leading term yields the familiar quadratic envelope [5]

$$\frac{\tau}{\tau_w} \simeq 1 + \left(\frac{\Delta s}{\sqrt{2}\,\tau_w}\right)^2. \tag{6}$$

A much better approximation can be obtained by keeping up to the second order of $\tau/\tau_w - 1$ in Δs^2, which yields

$$\frac{\tau}{\tau_w} \simeq 3\sqrt{1 + \left(\frac{\Delta s}{\sqrt{3}\,\tau_w}\right)^2} - 2. \tag{7}$$

Setting $\Delta s = s - s_w$ with

$$s_w = \pm\,\tau_w\int_1^{\tau_0/\tau_w} \frac{dx}{\sqrt{2\ln x}} \simeq \pm\,\tau_w\sqrt{\frac{(\tau_0/\tau_w + 2)^2}{3} - 3}\,, \tag{8}$$

the origin shifts from the waist to arbitrary initial values τ_0 and τ_0', where the sign is chosen according to the opposite sign of τ_0'. Figure 1 shows the beam spreading

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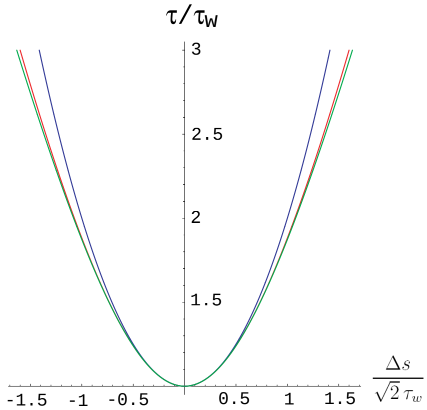

FIG. 1: Beam spreading curve in drift space. The red line is the exact result. The blue line is the quadratic approximation. The green line is the better approximation given in Eq. (7).

curve and the approximations given in Eqs. (6) and (7). In a typical split photoinjector, the maximum beam size in the drift is several times the size of the beam waist, thus Eq. (7) is more adequate to use.

The beam spreading discussed above describes approximately the evolution of each longitudinal slice in the drift space. Due to variations in slice perveance, initial position, and initial momentum, etc., each slice has a slightly different envelope determined by their initial values of τ_0 and τ_0'. From the envelopes of all the slices, bunch emittance can be computed. To avoid the unknown slice perveance distribution over the bunch, we will work in the τ-space instead of the $\hat\sigma$-space, which is sufficient for understanding the emittance oscillation.

III. EMITTANCE CALCULATION FORMULA

To obtain the rms emittance of a bunch, the following expression needs to be calculated:

$$\epsilon = \sqrt{\overline{X^2}\,\overline{P^2} - \overline{XP}^{\,2}}, \tag{9}$$

where X and P are coordinate and momentum, respectively. An overbar means averaging over the particles. For our purpose here, X and P stand for $\hat\sigma$ and $\hat\sigma'$, or τ and τ'. Although easy to compute numerically, this emittance expression is rather difficult to manipulate analytically. Here we derive an emittance formula assuming that X and P vary around their averages, depending linearly on a set of uncorrelated small deviations q^α in some variables (initial conditions, fluctuating parameters, etc.), i.e.,

$$\begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} \bar X \\ \bar P \end{pmatrix} + \sum_\alpha \begin{pmatrix} \partial_{q^\alpha}\bar X \\ \partial_{q^\alpha}\bar P \end{pmatrix} q^\alpha , \tag{10}$$

with $\bar q^\alpha = 0$ and $\overline{q^\alpha q^\beta} = 0$ for $\alpha \neq \beta$. Then

$$\begin{aligned}
\epsilon^2 &= \left|\overline{\begin{pmatrix} X \\ P \end{pmatrix}(X,P)}\right| \\
&= \left|\begin{pmatrix} \bar X \\ \bar P \end{pmatrix}(\bar X,\bar P) + \sum_{\alpha,\beta}\begin{pmatrix} \partial_{q^\alpha}\bar X \\ \partial_{q^\alpha}\bar P \end{pmatrix}(\partial_{q^\alpha}\bar X, \partial_{q^\alpha}\bar P)\,\overline{q^\alpha q^\beta}\right| \\
&= \left|\begin{pmatrix} \bar X \\ \bar P \end{pmatrix}(\bar X,\bar P) + \sum_{\alpha}\begin{pmatrix} \partial_{q^\alpha}\bar X \\ \partial_{q^\alpha}\bar P \end{pmatrix}(\partial_{q^\alpha}\bar X, \partial_{q^\alpha}\bar P)\,\overline{(q^\alpha)^2}\right| \\
&= \begin{vmatrix} \hat X\cdot\hat X & \hat X\cdot\hat P \\ \hat X\cdot\hat P & \hat P\cdot\hat P \end{vmatrix} \\
&= \left\|\hat X \times \hat P\right\| = \sum_{\alpha<\beta}\left(\hat X_\alpha \hat P_\beta - \hat X_\beta \hat P_\alpha\right)^2 \\
&= \sum_\alpha \left(\bar X\partial_{q^\alpha}\bar P - \bar P\partial_{q^\alpha}\bar X\right)^2 \overline{(q^\alpha)^2} + O\left(\overline{(q^\alpha)^2}\,\overline{(q^\beta)^2}\right),
\end{aligned} \tag{11}$$

where $\hat X$ is the vector $(\bar X, (\partial_{q^1}\bar X)q^1_{\rm rms}, \cdots)$, and similarly for $\hat P$. A pair of vertical bar stands for the determinant. Note that the second-to-last step is the *Lagrange's Identity*, which gives the residual of the well-known *Cauchy-Schwarz inequality.*

Now let us connect this emittance formula to expressions used in the literature. Apply this formula to the emittance of a bunch whose slices' envelopes $\hat\sigma$ and $\hat\sigma'$ depend on current; then we have

$$\epsilon = \left|\hat\sigma\partial_I\hat\sigma' - \hat\sigma'\partial_I\hat\sigma\right|_{I_p}\widehat{\delta I} = \hat\sigma(I_p)^2\left|\frac{\partial}{\partial I}\left(\frac{\hat\sigma'}{\hat\sigma}\right)\right|_{I_p}\widehat{\delta I}, \tag{12}$$

where $\widehat{\delta I}$ stands for the standard deviation from I_p. This is the expression given in Eq. (2.8) of [3] (except for minor differences that might be due to typos or different definitions). Another commonly used expression is the two-slice emittance

$$\epsilon = \frac{1}{2}\left|\,\hat\sigma_+\hat\sigma_-' - \hat\sigma_-\hat\sigma_+'\right| = \frac{\beta\gamma}{2}\left|\,\sigma_+\sigma_-' - \sigma_-\sigma_+'\right|, \tag{13}$$

which is the same as the previous expression provided that we let $\hat\sigma_+ = \hat\sigma_{I_p}$ and $\hat\sigma_- = \hat\sigma_{I_p} + \partial_I\hat\sigma\,\Delta I$, i.e., $\partial_I\hat\sigma = (\hat\sigma_- - \hat\sigma_+)/\Delta I$ and $\widehat{\delta I} = \Delta I/2$.

IV. EMITTANCE OSCILLATION IN THE DRIFT OF SPLIT PHOTOINJECTORS

In τ-space, the envelopes of bunch slices differ slightly due to small variations in their initial values τ_0 and τ_0'. Applying the emittance formula, we can write

$$\epsilon \simeq \sqrt{W_\tau^2\,\frac{(\Delta\tau_0)^2_{\rm std.}}{(\tau_0)^2_{\rm avg.}} + W_{\tau'}^2\,\frac{(\Delta\tau_0')^2_{\rm std.}}{(\tau_0')^2_{\rm avg.}}}\,, \tag{14}$$

where

$$W_\tau = \left(\tau\partial_{\tau_0}\tau' - \tau'\partial_{\tau_0}\tau\right)\tau_0\,, \quad W_{\tau'} = \left(\tau\partial_{\tau_0'}\tau' - \tau'\partial_{\tau_0'}\tau\right)\tau_0'.$$

The quantities in parentheses need to be evaluated for the average envelope.

To calculate the two derivatives in W_τ, we take the partial derivative of Eq. (5) with respect to τ_0, which gives

$$-\frac{s}{\tau_w^2}\frac{\partial \tau_w}{\partial \tau_0}+\frac{1}{\tau_0'}\frac{\partial(\tau_0/\tau_w)}{\partial \tau_0}=\frac{1}{\tau'}\frac{\partial(\tau/\tau_w)}{\partial \tau_0}. \quad (15)$$

On the left-hand side, the first term equals $-s/\tau_w\tau_0$ and the second term vanishes. On the right-hand side, we can either write it as a function of $\partial_{\tau_0}\tau$ or replace τ/τ_w with $e^{\tau'^2/2}$ to get a function of $\partial_{\tau_0}\tau'$. These lead to the necessary expressions

$$\partial_{\tau_0}\tau=\frac{\tau-\tau' s}{\tau_0} \quad \text{and} \quad \partial_{\tau_0}\tau'=-\frac{s}{\tau_0\tau}. \quad (16)$$

Therefore, W_τ can be written as

$$W_\tau=(\tau'^2-1)s-\tau\tau', \quad (17)$$

which is a function of the envelope of the central slice.

Similarly, we have

$$\partial_{\tau_0'}\tau=\tau_0\tau'-\tau_0'(\tau-\tau' s) \quad \text{and} \quad \partial_{\tau_0'}\tau'=\frac{\tau_0+\tau_0' s}{\tau}, \quad (18)$$

and $W_{\tau'}$ can be written as

$$W_{\tau'}=\tau_0\tau_0'(1-\tau'^2)-\tau_0'^2 W_\tau. \quad (19)$$

Equations (14), (17), and (19), and the solution for τ provide an analytical expression for the emittance.

Figure 2 illustrates the functions W_τ and $W_{\tau'}$, their absolute values, as well as the root of their square sum. We see that $|W_\tau|$ and $|W_{\tau'}|$ start at the same value since $W_\tau(0)=-W_{\tau'}(0)=-\tau_0\tau_0'$. Furthermore, $|W_\tau|$ has two minima while $|W_{\tau'}|$ has one. From $W_\tau'=2[(\tau'/\tau)s-1]=0$, it is clear that the function W_τ has a minimum $W_\tau^{\min}=-s$ at $s=\tau/\tau'$. Since $\tau_0>0$ and $\tau_0'<0$, W_τ starts from a positive value, reaches a negative minimum value, then becomes positive again and approaches $\tau'^2 s$. Therefore, $|W_\tau|$ generally has two minima where W_τ crosses zero and one local maximum that is proportional to s. In other words, $|W_\tau|$ has a generic W-shaped oscillation with double minima, a feature of emittance oscillation observed in the drift of split photoinjectors. Other the other hand, $W_{\tau'}$ starts from a negative value and crosses zero once, thus $|W_{\tau'}|$ only has one minimum.

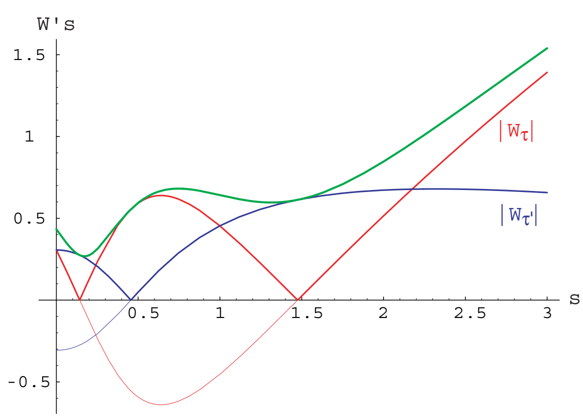

FIG. 2: Functions W_τ in red and $W_{\tau'}$ in blue, with their absolutes in heavier stroke. The green curve is the root of square sum of these two functions. The initial τ_0 and τ_0' are 0.47 and -0.65, respectively.

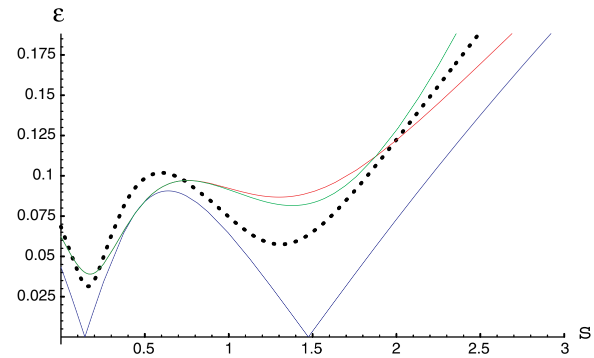

FIG. 3: Emittance (in τ-space) oscillation in the drift of a SPARC photoinjector design. The dotted line is the numerical tracking result. The blue line is the contribution from τ_0 variation only, which is proportional to $|W_\tau|$. The red line includes contributions from both τ_0 and τ_0' variations. The green line is computed using the approximate envelope solution given in Eq. (7).

As a concrete example, we show the emittance oscillation in the drift of a SPARC photoinjector design, which is similar to the LCLS design. Figure 3 shows the result. The blue curve is the τ-space emittance due to τ_0 variations that is proportional to $|W_\tau|$, which has two minima at zero crossing as discussed above. With the additional contribution from $W_{\tau'}$, the minima are raised from zeros and smoothed. The total emittance based on Eq. (14) is the red line, which agrees reasonably well with the numerical tracking result (the dots). The difference could be due to the neglected correlation (with a coefficient 0.23) between variations in τ_0 and τ_0'. This example demonstrates our understanding of the “double-minimum” feature in the emittance oscillation.

It is, however, clearly possible to have only one minimum in the emittance oscillation in the case where the variation in τ_0' (and thus $W_{\tau'}$) is more significant. For example, if the τ_0 variation in the above example is four times smaller, the emittance oscillation will be as shown in Fig. 4. Such a behavior could be those observed in split photoinjectors with an ellipsoidal bunch profile [6].

Finally we remark that the matching point into a booster should be at the beam waist, where $\sigma'=0$, a condition required by the invariant envelopes. The local emittance maximum in general does not have to be located at the beam waist.

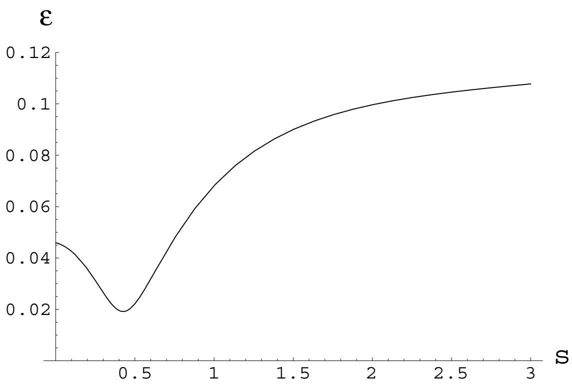

FIG. 4: Emittance oscillation with τ_0 variations four times smaller than in the previous SPARC example.

V. ACKNOWLEDGEMENT

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[1] M. Ferrario, J.E. Clendenin, D.T. Palmer, J.B. Rosenzweig, L. Serafini, SLAC-PUB-8400 (2000).
[2] M. Ferrario et al., Proceedings of the ICFA Workshop on the Physics and Applications of High Brightness Electron Beams, Sardinia, Italy, July 2002.
[3] L. Serafini and J.B. Rosenzweig, Phys. Rev. E **55**(6), 7565 (1997).
[4] See, for example, J.D. Lawson, *The Physics of Charged Particle Beams*, 2nd ed. (Oxford University Press, New York, 1988).
[5] M. Reiser, *Theory and design of charged particle beams*, (Wiley & Sons, Inc., New York, 1994).
[6] K. Flöttmann and C. Limborg, private communication at the PAHBEB2005 workshop.